Wave Motion and Sound I

Wave : Mechanical Wave — motion of massive string, water surface
— sound wave . . .

Electromagnetic wave — light, microwave

Gravitational wave — variation of spacetime

Mechanical Wave

			e.g	
① transverse	up/down	vibration	S-wave	~3000 m/s
② longitudinal	back/fro	vibration	P-wave	~5000 m/s.

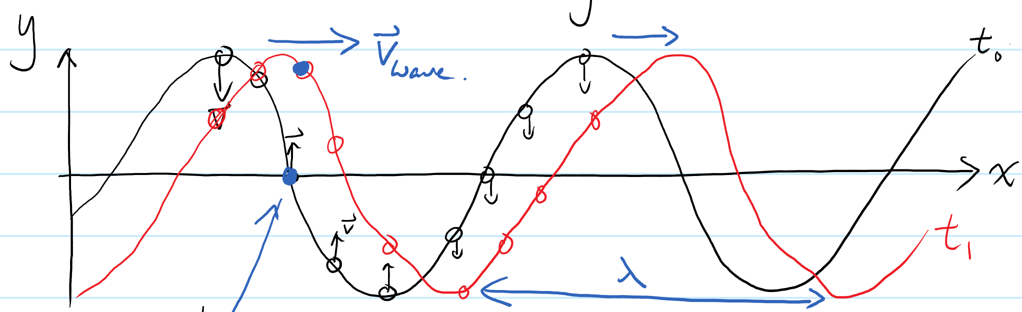
$\boxed{\text{No matters/ particles actually propagate}}$

only the "phase" is propagating.

e.g human wave

Travelling Wave (Sinusoidal wave, continuous)

Particles move collectively

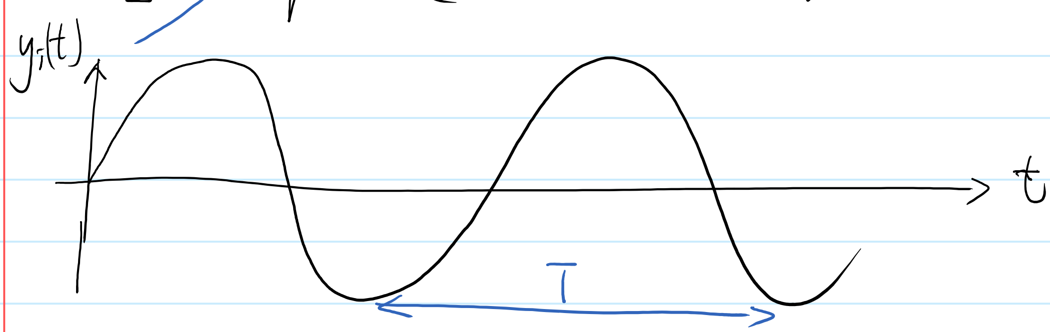


snapshot.

Sin (phase)

same phase = { same position
same $\vec{v}$.

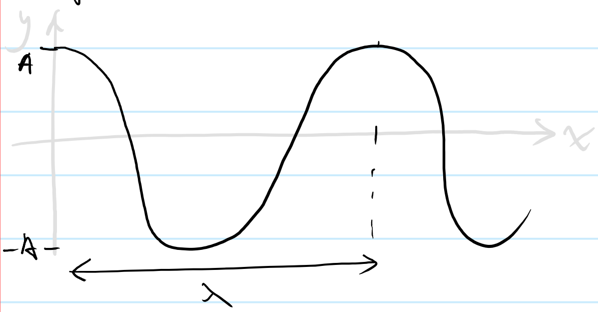
Each particle exhibits SHM.



phase	time	x
2π	T	λ

Mathematical Description

in fixed time



Let $y = A\cos(kx)$

to satisfy $x = 0$, $y = A$

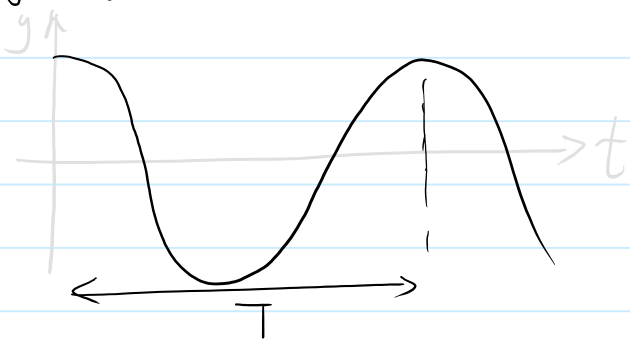
$x = \lambda$, $y = A$

$\Rightarrow \cos(k\lambda) = 1$

$\Rightarrow k\lambda = 2\pi$

$\boxed{k = \frac{2\pi}{\lambda}}$ wave number

for fixed x



$y = A\cos(\omega t)$

for $t = 0$ & $t = T$, $y = A$.

$\Rightarrow \omega t = 2\pi \Rightarrow \boxed{\omega = \frac{2\pi}{T} = 2\pi f}$

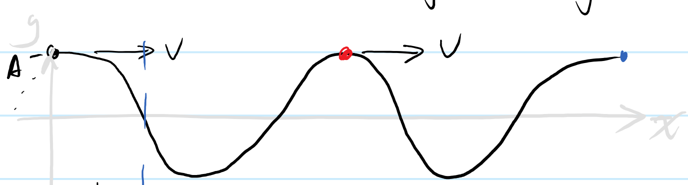
angular frequency (rad/s)

Wave function

$y(x, t)$ at time t the mass at x has a displacement y.

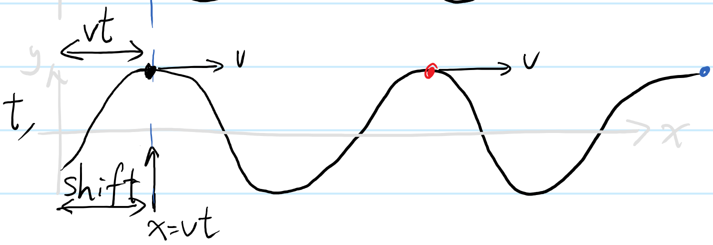
How to write down a wavefunction for a sinusoidal wave?

At $t = 0$,



$y(x,0) = A\cos(kx)$, $k = \frac{2\pi}{\lambda}$

At some t, $t > 0$



$y(x,t) = A\cos(kx + \phi)$

shift in cos

at $x = vt$ $y = A \Rightarrow \cos(kx + \phi) = 1 \Rightarrow kx + \phi = 0$ first max.

$\Rightarrow \phi = -kx = -kvt$

Therefore, a wavefunction at any time t is

$$y(x,t) = A\cos(kx - kvt)$$

Since every part of the wave exhibits SHM, taking $x=0$ as example

$$y(0,t) = A\cos(-kvt) = \overset{\text{SHM}}{A\cos(\omega t)} = A\cos(-\omega t)$$

$$\Rightarrow \quad \omega = kv$$

$$\Rightarrow \quad 2\pi f = \frac{2\pi}{\lambda} v$$

$$\Rightarrow \quad \boxed{v = f\lambda.} = \frac{\omega}{k}.$$

Finally, we write

relatively opposite sign.

$$y(x,t) = \underline{A}\cos(kx - \omega t + \underline{\phi_0})$$

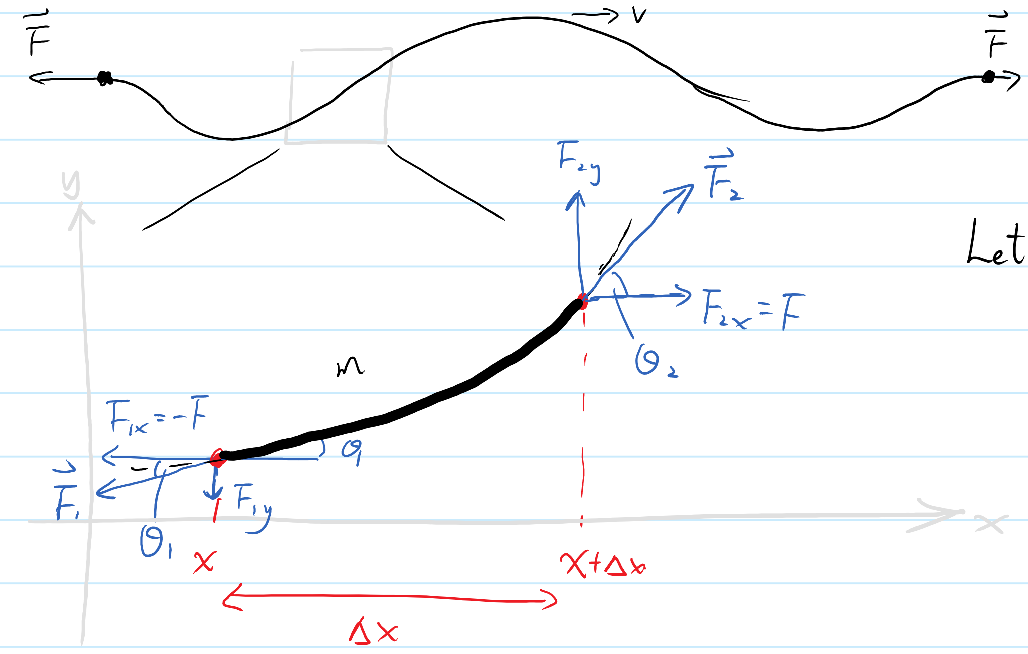
defined by initial conditions

for a wave travelling to the $+x$ direction with speed $v = \frac{\omega}{k}$.

For a wave travelling to the $-x$-direction $v \to -v$, $\phi = kvt$
$= +\omega t$

$$y(x,t) = A\cos(\underbrace{kx + \omega t}_{\text{relatively same sign.}} + \phi_0)$$

Wave Equation (string)



Let μ = mass per unit length

F = Tension

The segment of string has a mass $m = \mu \cdot \Delta x$

The segment accelerates only vertically but not horizontally.

$$\Rightarrow \quad a_x = 0 \quad \text{as} \quad F_{1x} = -F = -F_{2x}.$$

For vertical motion, $\quad F_{net_y} = m a_y$

$$F_{1y} + F_{2y} = (\mu \Delta x)\, a_y$$

$$-F\tan\theta_1 + F\tan\theta_2 = \mu \Delta x\, a_y$$

$\tan\theta$ = slope $= \dfrac{\partial y}{\partial x}$.

$$-F\left(\left.\frac{\partial y}{\partial x}\right|_x - \left.\frac{\partial y}{\partial x}\right|_{x+\Delta x}\right) = \mu \Delta x\, a_y$$

$a_y = \dfrac{\partial^2 y}{\partial t^2}$

partial derivative $\because$ y is function of 2 variables.

$$\underbrace{\frac{\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_x}{\Delta x}} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}$$

for $\Delta x \to 0$,

$$\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}$$

$\Rightarrow$ Wave Eq$^{\underline{n}}$. $$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}}$$

Why is it called wave eq$^{\underline{n}}$? Because function behaves as a wave satisfies the eq$^{\underline{n}}$.

Try $y(x,t) = A\cos(kx-\omega t)$

$$\frac{\partial y}{\partial x} = -kA\sin(kx-\omega t) \quad , \quad \frac{\partial^2 y}{\partial x^2} = -k^2 A\cos(kx-\omega t)$$

$$\frac{\partial y}{\partial t} = +\omega A\sin(kx-\omega t) \quad , \quad \frac{\partial^2 y}{\partial t^2} = -\omega^2 A\cos(kx-\omega t)$$

$$LHS = \frac{\partial^2 y}{\partial x^2} \quad \Bigg| \quad RHS = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2}$$

$$= -k^2 A\cos(kx-\omega t) \quad \Bigg| \quad = \frac{\mu}{F}(-\omega^2)A\cos(kx-\omega t)$$

LHS = RHS iff. $k^2 = \frac{\mu}{F}\omega^2 \Rightarrow \frac{F}{\mu} = \frac{\omega^2}{k^2} = V^2$

$\Rightarrow$ $y(x,t)$ satisfies the wave eq$\underline{n}$. iff. $V = \frac{\omega}{k}$.

In general, a wave eq$^{\underline{n}}$ has a form.

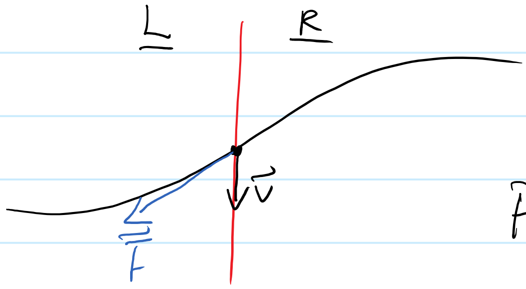
$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2 y}{\partial t^2}}$$

where V is the wave propagation speed.

i.e. for a string the speed of the wave is

$V = \sqrt{\frac{F}{\mu}}$ depends on the properties of the string, but not how the wave is generated.

Power

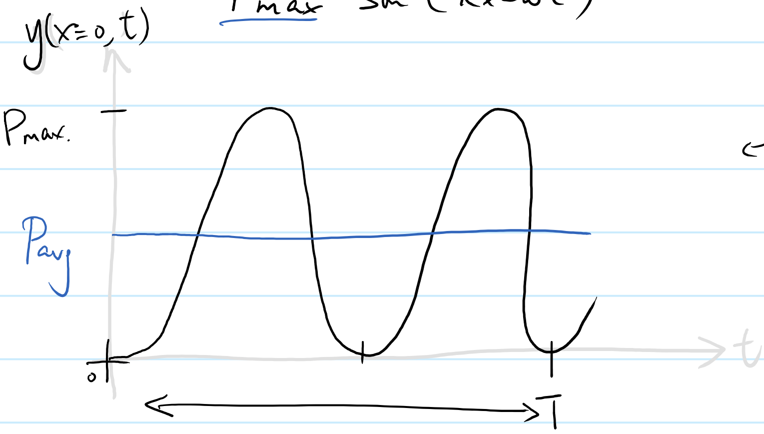


work done on Right side by Left side / time

$$P = \vec{F}\cdot\vec{v}$$
$$= F_y v_y$$
$$= -F\tan\theta \cdot v_y \qquad y = A\cos(kx-\omega t)$$
$$= -F\cdot\frac{\partial y}{\partial x}\cdot\frac{\partial y}{\partial t}$$
$$= -F\left[-kA\sin(kx-\omega t)\right]\cdot\left[\omega A\sin(kx-\omega t)\right]$$

$$P = Fk\omega A^2\sin^2(kx-\omega t)$$
$$= \underbrace{\sqrt{\mu F}\,\omega^2 A^2}\sin^2(kx-\omega t) \geq 0$$ ↑ consistent with right-going wave.
$$\equiv P_{max}\sin^2(kx-\omega t)$$

take $x=0$ as example.



Average Power

$$P_{avg} = \frac{1}{T}\int_0^T P(x,t)\,dt \qquad \text{time average.}$$
$$= \frac{1}{T}\int_0^T P_{max}\sin^2(kx-\omega t)\,dt$$
$$= \frac{1}{T}P_{max}\int_0^T \underbrace{\frac{1}{2}}_{=T/2}\Big[1-\underbrace{\cos(2kx-2\omega t)}_{=0 \text{ after integration}}\Big]dt$$

$$\Rightarrow \boxed{P_{avg} = \frac{P_{max}}{2}}$$

Wave Reflection

General properties of wave.

① Superposition of wave: if y_1, y_2 are solutions of the wave eq^n.

then $y = y_1 + y_2$ is also a solution.

② Obey boundary condition: physical constrains the wave must satisfies.

e.g.

The wave must have $y=0$ at $x=0$ for all time t.

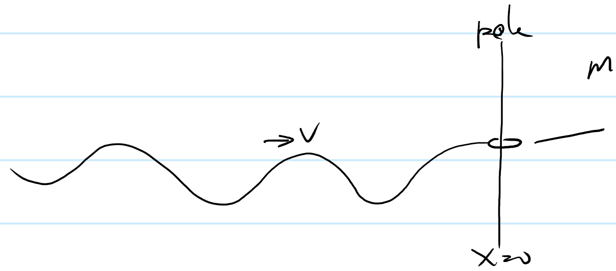


Image method (see note for examples & diagrams)

① Suppose a wave going towards the wall/pole is y_1

② Imagine there is a wave coming from behind the wall / pole.
$\hookrightarrow y_2$

③ we know $y_1 + y_2$ must satisfy the boundary condition

- for wall, $y_1 + y_2 = 0$ at $x=0$ for $\forall t$.
- for pole, slope of $(y_1 + y_2) = 0$ at $x=0$ for $\forall t$.

④ Once a suitable y_2 is found, we identify y_2 as the reflection of y_1.